Dispersion Characteristics of Microstrip Lines

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Abstract—**A coupled-line analysis is used in conjunction with Carlin's model in order to find the frequency dependence of the propagation constant and of the characteristic impedance of dispersive microstrip lines. The propagation constant thus obtained is identical with that of Carlin and the characteristic impedance of the coupled lines decreases with frequency.**

I. Introduction

THERE HAVE been numerous quasi-static analyses of microstrip lines, but only in the last decade has attention been given to the dispersive behavior of these lines at higher frequencies [1]–[11]. In this paper, a coupled-line analysis [1] is used in conjunction with Carlin's model [6] in order to obtain the frequency dependence of the propagation constant and of the characteristic impedance of dispersive microstrip lines. The frequency variation of the propagation constant is well established while that of the characteristic impedance which has been reviewed by Bianco *et al.* [9], is related to its definition. According to some authors, the characteristic impedance increases with frequency [2], [3], [7], [8] while according to others, it decreases with frequency [1], [10]. In our coupled-line analysis, the characteristic impedance of the lines defined in accordance with a propagation constant whose value has been experimentally verified, is found to decrease with frequency.

II. The Carlin Model

A coupled-line model for dispersion in microstrips has been proposed by Carlin [6]. This model is based on the circuit properties of coupled dispersive transmission lines as given by Noble and Carlin [13]. Let the infinitely long uniform microstrip extend in the z-direction. The equations for the pair of coupled lines representing it, become, for real frequencies ω

$$\frac{d}{dz}\begin{bmatrix}\bar{v}\\ \bar{i}\end{bmatrix} = -\begin{bmatrix}0 & \bar{\bar{Z}}\\ \bar{\bar{Y}} & 0\end{bmatrix}\begin{bmatrix}\bar{v}\\ \bar{i}\end{bmatrix}$$

where

$$\bar{v}=\begin{bmatrix}v_1\\ v_2\end{bmatrix}$$

and

$$\bar{i}=\begin{bmatrix}i_1\\ i_2\end{bmatrix}$$

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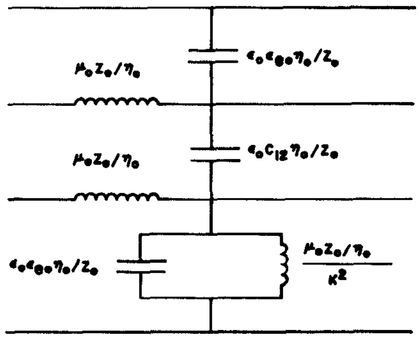

Fig. 1. Coupled-line model per unit length for microstrip.

are the voltage and current vectors on the pair of coupled lines, while $\bar{\bar{Z}}$ and $\bar{\bar{Y}}$ are the series impedance per unit length and the shunt admittance per unit length two-by-two matrices for the pair of coupled lines. $\bar{\bar{Z}}$ and $\bar{\bar{Y}}$ are given by [6]

$$\bar{\bar{Z}}=\begin{bmatrix}Z_{11} & Z_{12}\\ Z_{12} & Z_{22}\end{bmatrix}=\begin{bmatrix}j\omega\mu_0\dfrac{Z_0}{\eta_0} & 0\\ 0 & j\omega\mu_0\dfrac{Z_0}{\eta_0}\end{bmatrix}$$

$$\bar{\bar{Y}}=\begin{bmatrix}Y_{11} & Y_{12}\\ Y_{12} & Y_{22}\end{bmatrix}=\begin{bmatrix}j\omega\epsilon_0\epsilon_{e0}\dfrac{\eta_0}{Z_0} & j\omega\epsilon_0 C_{12}\dfrac{\eta_0}{Z_0}\\ j\omega\epsilon_0 C_{12}\dfrac{\eta_0}{Z_0} & j\omega\epsilon_0\epsilon_{e0}\dfrac{\eta_0}{Z_0}+\dfrac{K^2}{j\omega\mu_0\dfrac{Z_0}{\eta_0}}\end{bmatrix}$$

where ϵ_0, μ_0, and η_0 are the permittivity, permeability, and impedance of free space, ϵ_{e0} and Z_0 are the effective static relative dielectric constant and the air-filled static microstrip line characteristic impedance, respectively, K is the cutoff wavenumber for the uncoupled TE mode (see below), and the coupling capacitance $C_{12}=k\epsilon_{e0}$ (where $0<k<1$ is the capacitive coefficient of coupling). None of these quantities is frequency-dependent so that all of the above matrix elements are linearly dependent on the frequency, except Y_{22}. It is to be noted that $\epsilon_{e0}=(Z_0/Z_d)^2$ where Z_d is the dielectric-filled static microstrip line characteristic impedance. The coupled-line model for the microstrip is shown in Fig. 1. We have normalized Carlin's expressions for the $\bar{\bar{Z}}$ and $\bar{\bar{Y}}$ matrices for reasons which will appear in the next section.

When uncoupled, the upper line propagates a TEM mode while the lower one propagates a TE mode. When

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coupled, the circuit represents a pair of modes which are the two lowest order hybrid modes that propagate on the microstrip line. The eigenvalues $\gamma_{a,b}^2$ of the $\begin{bmatrix} 0 & \overline{\overline{Z}} \\ \overline{\overline{Y}} & 0 \end{bmatrix}$ matrix are given by

$$\gamma_{a,b}^2 = -\omega^2\epsilon_0\mu_0\epsilon_{e0} + \frac{K^2}{2} \mp \sqrt{k^2\omega^4\epsilon_0^2\mu_0^2\epsilon_{e0}^2 + \frac{K^4}{4}}$$

where the minus sign in front of the square root relates to the mode which propagates down to dc a, i.e., the quasi-TEM mode. The plus sign relates to the next mode b, i.e., the quasi-TE_{10} mode.

III. The Coupled Line Analysis

Tripathi [12] has analyzed asymmetric coupled transmission lines in an inhomogeneous medium. The behavior of the two infinitely long coupled lines, which extend in the z-direction, is described by the following set of equations:

$$\frac{d}{dz}\begin{bmatrix} v_1 \\ v_2 \\ i_1 \\ i_2 \end{bmatrix} = -\begin{bmatrix} 0 & 0 & Z_1 & Z_m \\ 0 & 0 & Z_m & Z_2 \\ Y_1 & Y_m & 0 & 0 \\ Y_m & Y_2 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ i_1 \\ i_2 \end{bmatrix}$$

where v_i and i_i are the voltage and current on line i ($i=1,2$), while Z_j and Y_j ($j=1,2$) are the self impedance and admittance, respectively, per unit length of line j in the presence of line k ($k=1,2$; $k\neq j$); Z_m and Y_m are the mutual impedance and admittance, respectively, per unit length of the coupled lines. None of the Z or Y coefficients vary with z, so that a z-variation of the form $e^{\gamma z}$ could be assumed for the voltages and currents, resulting in a fourth-order characteristic equation whose roots (the matrix eigenvalues) are given by $\pm\gamma_a$ and $\pm\gamma_b$, where [12]

$$\gamma_{a,b}^2 = \frac{Y_1Z_1 + Y_2Z_2}{2} + Y_mZ_m \mp \frac{1}{2} \cdot \sqrt{(Y_1Z_1 - Y_2Z_2)^2 + 4(Z_1Y_m + Y_2Z_m)(Z_2Y_m + Y_1Z_m)}\ .$$

The above expression is identical to that given in [6], if we take $Z_1 = Z_{11}$, $Z_m = Z_{12}$, $Z_2 = Z_{22}$, $Y_1 = Y_{11}$, $Y_m = Y_{12}$, and $Y_2 = Y_{22}$. We should note that $Z_1 = Z_2$ and $Z_m = 0$. The microstrip could therefore be analyzed according to [12].

Following Tripathi's analysis, we assume then that the general solution for the voltages and currents on the two coupled lines (1 and 2) are given in terms of the four waves by

$$\begin{bmatrix} v_1 \\ v_2 \\ i_1 \\ i_2 \end{bmatrix} = A_1\begin{bmatrix} 1 \\ R_a \\ Y_{a1} \\ R_aY_{a2} \end{bmatrix} e^{-\gamma_a z} + A_2\begin{bmatrix} 1 \\ R_a \\ -Y_{a1} \\ -R_aY_{a2} \end{bmatrix} e^{+\gamma_a z} + A_3\begin{bmatrix} 1 \\ R_b \\ Y_{b1} \\ R_bY_{b2} \end{bmatrix} e^{-\gamma_b z} + A_4\begin{bmatrix} 1 \\ R_b \\ -Y_{b1} \\ -R_bY_{b2} \end{bmatrix} e^{+\gamma_b z}$$

where

$$R_{a,b} = \frac{1}{2Y_m}\left[Y_2 - Y_1 \mp \sqrt{(Y_2 - Y_1)^2 + 4Y_m^2}\ \right]$$

are real quantities, and where the coupled lines 1 and 2 characteristic admittances for modes a and b are given, respectively, by

$$Y_{a1,b1} = \frac{\gamma_{a,b}}{Z_1}$$

$$Y_{a2,b2} = \frac{\gamma_{a,b}}{Z_2}.$$

It follows that $Y_{a1} = Y_{a2} \triangleq Y_a$ and $Y_{b1} = Y_{b2} \triangleq Y_b$ since $Z_1 = Z_2$. The quantities in the square brackets multiplying the exponential terms are the four eigenvectors corresponding to the four eigenvalues $\mp\gamma_{a,b}$.

The accuracy of the fundamental mode eigenvalue γ_a has been experimentally verified [6]. The corresponding coupled lines (1 or 2) characteristic impedance is given by

$$Z_a = \frac{j\omega\mu_0 Z_0/\eta_0}{\gamma_a} = \frac{\gamma_a}{j\omega\epsilon_0\epsilon_e\eta_0/Z_0} = \frac{Z_0}{\sqrt{\epsilon_e}} = Z_d\sqrt{\frac{\epsilon_{e0}}{\epsilon_e}}$$

where ϵ_e is the effective frequency-dependent relative dielectric constant related to the propagation constant γ_a through $\gamma_a = j\omega\sqrt{\mu_0\epsilon_0\epsilon_e}$. The normalization of Carlin's matrices $\overline{\overline{Z}}$ and $\overline{\overline{Y}}$ leaves the γ's and the R's unmodified but ensures that when ω tends to zero, Z_a tends to Z_d. When ω tends to zero, R_a tends to $\omega^2\epsilon_0\mu_0C_{12}/K^2$, i.e., it is real, positive, and tends to zero, too.

The coupled lines (1 or 2) characteristic impedance of the b modes is given similarly by

$$Z_b = \frac{j\omega\mu_0 Z_0/\eta_0}{\gamma_b}.$$

It is a purely imaginary quantity when the b-mode is under cutoff. When ω tends to zero, R_b tends to $-K^2/\omega^2\epsilon_0\mu_0C_{12}$, i.e., it is real negative, and tends to minus infinity.

Y_a and Y_b are the eigenvalues of the coupled lines characteristic admittance matrix given by [14]

$$\overline{\overline{Y}}_c \triangleq \overline{\overline{Z}}^{-1}\sqrt{\overline{\overline{Z}}\,\overline{\overline{Y}}} = \frac{1}{R_a - R_b}\begin{pmatrix} R_aY_b - R_bY_a & Y_a - Y_b \\ Y_a - Y_b & R_aY_a - R_bY_b \end{pmatrix}.$$

IV. Discussion and Conclusions

We have shown that as long as Carlin's model gives valid results for the fundamental mode propagation constant, the dispersive microstrip line could be appropriately represented by "equivalent" coupled transmission lines having propagation constants and characteristic impedances as given above. If the b-mode is strongly attenuated under cutoff (which are the usual working conditions) or suppressed above cutoff (which seems difficult to achieve), the a-mode only will propagate on lines 1 and 2 and the dispersive microstrip line representation would be as shown in Fig. 2(a). We know that $v_2/v_1 = i_2/i_1 = R_a$. We

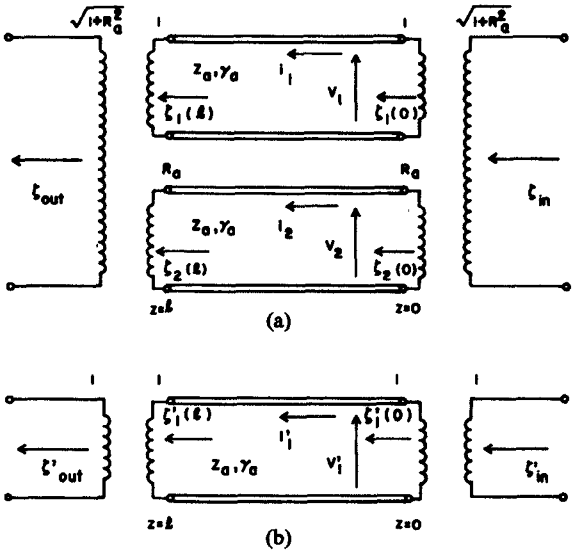

Fig. 2. (a) Dispersive microstrip line representation by two transmission lines. (b) Equivalent circuit consisting of one transmission line.

have omitted for convenience the suffix a from v and i. It follows that the impedances ($\zeta \triangleq v/i$) at the same point (z) on the two lines (1 and 2) are equal, i.e., $\zeta_1(z)=\zeta_2(z)$. At the output ($z=l$), it is required that $\zeta_1(l)=\zeta_2(l)=\zeta_{out}$ and at the input ($z=0$) it is required that $\zeta_1(0)=\zeta_2(0)=\zeta_{in}$. The power on line 1 is $1/2\,\mathrm{Re}\,v_1i_1^*$ while the power on line 2 is $1/2\,\mathrm{Re}\,v_1i_1^*R_a^2$, i.e., the total power in the circuit is $1/2\,\mathrm{Re}\,v_1i_1^*(1+R_a^2)$. In order to comply with these constraints we take at the output the ratios of the voltages (and of the currents) as follows: $v_1(l):v_2(l):v_{out}=1:R_a:\sqrt{1+R_a^2}$. Identical ratios are taken for the voltages (and for the currents) at the input. The relation between ζ_{in} and ζ_{out} is given by

$$\zeta_{in}=Z_a\frac{\zeta_{out}+jZ_a\tanh\gamma_a l}{Z_a+j\zeta_{out}\tanh\gamma_a l}.$$

The above circuit could be replaced by an equivalent circuit (as shown in Fig. 2(b)) consisting of a single transmission line with characteristic parameters γ_a and Z_a, voltage (v_1') and current (i_1') such that $1/2\,\mathrm{Re}\,v_1'i_1'^*=1/2\,\mathrm{Re}\,v_1i_1^*(1+R_a^2)$ and $\zeta_{in}'=\zeta_{in}$. Since it is well established [1]–[11] that ϵ_e increases with frequency, our analysis shows that the characteristic impedance Z_a decreases with frequency, a fact which has been confirmed experimentally by Napoli and Hughes [15]. Denlinger [1], Bianco *et al.* [16], and Getsinger [10], arrived at similar expressions for Z_a by different approaches. It is to be stressed that this analysis is possible only because $Z_1=Z_2$. Finally, when ω tends to zero, $R_a^2\ll 1$, and all the power is concentrated in the a mode in line 1 only, $v_1'=v_1$, $i_1'=i_1$, and $Z_a=Z_d$.

The 4×4 equivalent Z-matrix of a segment of line of length l could be found by inserting the adequate values given above for $\gamma_{a,b}$, $Z_{a,b}$, $R_{a,b}$ in the appropriate expression given by Tripathi [12]. A similar matrix has been found by Bianco *et al.* [17]. If the b-mode is well under cutoff this 4×4 matrix reduces to a 2×2 matrix linking the voltage and current at the two ends of line l, as for an ordinary transmission line having characteristic parameters γ_a and Z_a, thus

$$\begin{bmatrix} v_{in} \\ v_{out}\end{bmatrix}=\begin{bmatrix} Z_a\coth\gamma_a l & Z_a\operatorname{cosech}\gamma_a l \\ Z_a\operatorname{cosech}\gamma_a l & Z_a\coth\gamma_a l\end{bmatrix}\begin{bmatrix} i_{in} \\ -i_{out}\end{bmatrix}.$$

There is no voltage and current at either end of line 2 in this case.

In conclusion it seems that the concept of characteristic impedance is useful mainly at the frequencies where the b-mode is well under cutoff, or if the b-mode is suppressed. In this case the dispersive microstrip line could be represented by an equivalent transmission line having a frequency dependent propagation constant γ_a and a frequency dependent characteristic impedance Z_a which decreases with frequency.

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